



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

*A General Method of Obtaining the Finite Integral of any Rational Algebraic Function of  $x$ ; or Summing any Series of which such a Function is the General Term. By WILLIAM ORCHARD, Esq., Fellow of the Institute of Actuaries.*

LET  $u_x = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  be the given function, then, by a well known theorem,

$$u_x = u_0 + x \Delta u_0 + \frac{x(x-1)}{1 \cdot 2} \Delta^2 u_0 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \Delta^3 u_0 + \&c., \quad (\text{A})$$

the integral of which, since  $u_0, \Delta u_0, \Delta^2 u_0, \&c.$ , are constants, and

$$\sum \frac{x(x-1)(x-2)\dots(x-p)}{1 \cdot 2 \cdot 3 \dots (p+1)} = \frac{x(x-1)\dots x-(p+1)}{1 \cdot 2 \cdot 3 \dots (p+2)},$$

will be

$$\sum u_x = xu_0 + \frac{x(x-1)}{1 \cdot 2} \Delta u_0 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \Delta^2 u_0 + \&c.; \quad (\text{B})$$

which differs from (A) only by the coefficients of  $\Delta u_0, \Delta^2 u_0, \&c.$ , being shifted each one place to the left.

The labour of integration is then reduced to finding the constants  $u_0, \Delta u_0, \Delta^2 u_0, \&c.$ , which requires the computation of  $u_x$  for  $x=0=1=2=\dots=(n-1)=n$ ; these values differenced, as in the annexed form, will then determine such constants:

$$\begin{array}{c} u_0 \\ \Delta u_0 \\ u_1 \quad \Delta^2 u_0 \\ \Delta u_1 \quad \Delta^3 u_0 \\ u_2 \quad \Delta^2 u_1 \\ \Delta u_2 \\ u_x \end{array}$$

The computation of  $u_0, u_1, u_2, \&c.$ , will be very easy, as  $u_0 = a_0, u_1 =$  the algebraic sum of the coefficients, and the others requiring but trifling operations, as will appear by the examples, in which they are fully given; it will only be necessary to compute  $u_0, u_1, \dots u_{n-1}$ , since from these  $\Delta u_0, \Delta^2 u_0, \dots \Delta^{n-1} u_0$ , may be found,  $\Delta^n u_0$  being known to be  $= a_n \times n(n-1) \dots 2 \cdot 1$ .

If the coefficients of  $u_0, \Delta u_0, \&c.$ , in (B) be denoted by  $X_1, X_2, \&c.$  (they are the successive binomial coefficients for index  $x$ ),

$$\sum u_x = u_0 \cdot X_1 + \Delta u_0 \cdot X_2 + \Delta^2 u_0 \cdot X^3 + \&c. \quad (\text{C})$$

Example 1.  $u_x = x^2$ .

$$\begin{array}{rcc} 0 & 1 & \therefore \sum x^2 = 0 + \frac{x(x-1)}{1 \cdot 2} + 2 \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}. \\ 1 & & \end{array}$$

This particular case may be more compactly represented thus—

$$\sum x^2 = \frac{x(x-1)}{1 \cdot 2} \left( 1 + \frac{2(x-2)}{3} \right) = \frac{x(x-1)(2x-1)}{1 \cdot 2 \cdot 3}.$$

10                  *Method of Obtaining the Finite Integral, &c.*

Example 2.  $u_2 = x^3$ .

$$\begin{array}{r} 0 \\ 1 \quad 6 \sum x^3 = \frac{x(x-1)}{1 \cdot 2} + 6 \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + 6 \frac{x(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3 \cdot 4} \\ 7 \\ 8 \end{array}$$


---

Example 3.  $u_x = x^2 + 5x - 4$ .

$$\begin{array}{r} -4 \\ +6 \\ +2 \end{array} \sum u_x = -4x + 6 \frac{x(x-1)}{1 \cdot 2} + 2 \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}$$


---

Example 4.  $u_x = 4x^3 - 5x^2 - 14x + 56$   
 $+3 - 8 + 40 = u_2$

$$\begin{array}{r} +56 \\ -15 \\ +41 \quad +14 \\ -1 \end{array} \sum u_x = 56X_1 + 15X_2 + 14X_3 + 24X_4$$


---

$$\begin{array}{r} +40 \\ +15 \\ +20 \quad +96 \\ +111 \quad +222 \quad +126 \\ +333 \\ +464 \end{array} \sum u_x = 5X_1 + 15X_2 + 96X_3 + 126X_4 + 24X_5$$


---

Example 6.  $u_x = x^5 + 10x^4 - 7x^3 + 14x^2 + 9x - 24$   
 $+12 +17 +48 +105 +186 = u_2$

$$\begin{array}{r} -24 \\ +27 \quad +156 \\ +3 \quad +468 \\ +183 \quad +624 \\ +186 \quad +948 \\ +993 \quad +807 \\ +2879 \\ +3372 \end{array} \sum u_x = -24X_1 + 27X_2 + 156X_3 + 468X_4 + 480X_5 + 120X_6$$


---

The value of the algebraic expressions in the preceding examples for  $x=2=3$ , &c., has been found by Horner's method, which is this; write down the coefficients of the expression in their usual order, with a cypher as the coefficient of any missing term, multiply the highest coefficient by the value of  $x=h$ , adding the product to the second, and so on, to the end;

thus, if  $ah^3 + bh^2 + ch + d$  were required, the successive results found would be  $ah + b$ ,  $ah^2 + bh + c$ ,  $ah^3 + bh^2 + ch + d$ .

This process repeated, stopping each time at one term short of the preceding, until all are exhausted, gives the result of writing  $x + h$  for  $x$  in any algebraic function, which has been applied by Horner to the resolution of numerical equations.

In summing a series of which  $u_x$  is the general term,  $\sum u_{x+1}$  is required, which corresponds with increasing  $x$  by unit in  $X_1$ ,  $X_2$ , &c.

There are many occasions on which it is necessary to obtain  $\Delta u_x$ ,  $\Delta^2 u_x$ ,  $\Delta^3 u_x$ , &c.,  $u_x$  being an algebraic function.

When the degree of the expression is not higher than the third, perhaps the easiest way is to use the particular case of Taylor's Theorem ( $h=1$ ).

$$F(x+1) - Fx = \Delta Fx = F'x + \frac{Fx''}{1 \cdot 2} + \frac{Fx'''}{1 \cdot 2 \cdot 3} + \text{&c.};$$

but in more complex expressions it will be better to apply Horner's method, of which it is the most simple case: subtracting from the coefficients at the close of each operation, the coefficients as they stood at the commencement. An example will make this clear.

$$\begin{aligned} 4x^5 + 5x^4 - 7x^3 + 1x^2 + 15x - 14 &= u_x \\ + 9 &+ 2 + 3 + 18 + 4 \\ + 13 &+ 15 + 18 + 36 \\ + 17 &+ 32 + 50 \\ + 21 &+ 53 \\ + 25 & \\ \hline + 20x^4 + 60x^3 + 49x^2 + 21x + 18 &= (u_{x+1} - u_x) = \Delta u_x \\ + 80 &+ 129 + 150 + 168 \\ + 100 &+ 229 + 279 \\ + 120 &+ 349 \\ + 140 & \\ \hline + 80x^3 + 300x^2 + 358x + 150 &= (\Delta u_{x+1} - \Delta u_x) = \Delta^2 u_x \\ + 380 &+ 738 + 888 \\ + 460 &+ 1198 \\ + 540 & \\ \hline + 240x^2 + 840x + 738 &= \Delta^3 u_x \\ + 1080 &+ 1818 \\ + 1320 & \\ \hline + 480x + 1080 &= \Delta^4 u_x \\ + 480 &= \Delta^5 u_x \end{aligned}$$

[The Calculus of Finite Differences is coming daily into so much greater requisition in investigations of the Theory of Life Assurance, that we willingly insert Mr. Orchard's Paper, although it has no other immediate reference to the subjects usually treated of in this Magazine.—Ed. A. M.]